

# On the zeros of orthogonal polynomials on the unit circle

María Pilar Alfaro<sup>1</sup>

*Departamento de Matemáticas, Universidad de Zaragoza,  
Calle Pedro Cerbuna s/n, 50009 Zaragoza, Spain*

Manuel Bello-Hernández<sup>1,\*</sup>

*Dpto. de Matemáticas y Computación, Universidad de La Rioja,  
Edif. J. L. Vives, Calle Luis de Ulloa s/n,  
26004 Logroño, Spain*

Jesús María Montaner<sup>1</sup>

*Departamento de Matemática Aplicada, Universidad de Zaragoza,  
Edificio Torres Quevedo, Calle María de Luna 3, 50018 Zaragoza, Spain*

---

## Abstract

Let  $\{z_n\}$  be a sequence in the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . It is known that there exists a unique positive Borel measure in the unit circle  $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$  such that the orthogonal polynomials  $\{\Phi_n\}$  satisfy

$$\Phi_n(z_n) = 0$$

for each  $n = 1, 2, \dots$ . Characteristics of the orthogonality measure and asymptotic properties of the orthogonal polynomial are given in terms of asymptotic behavior of the sequence  $\{z_n\}$ . Particular attention is paid to periodic sequence of zeros  $\{z_n\}$  of period two and three.

*Keywords:* Orthogonal polynomials, varying measures, zeros, asymptotics.  
*2000 MSC:* Primary 42C05; Secondary 33C47.

---

## 1. Introduction

A figure which displays the zeros of an orthogonal polynomial on the unit circle (OPUC) lets us state some properties of the Verblunsky coefficients and

---

\*Corresponding author

*Email addresses:* `palfaro@unizar.es` (María Pilar Alfaro), `mbello@unirioja.es` (Manuel Bello-Hernández), `montaner@unizar.es` (Jesús María Montaner)

<sup>1</sup>This research was supported in part from ‘Ministerio de Ciencia y Tecnología’, Project MTM2009-14668-C02-02

other parameters of OPUC (see Figures 1–4 and Section 8.4 of [19]). The zeros of OPUCs are eigenvalues of many operators. So, conclusions on the measure and other properties of OPUCs in terms of information about the zeros are interesting.

In the last decade several papers on zeros of OPUC have been published. For instance, we have [11], [12], [21], [22] and [23]. These articles joint to [3], [13], [14], [15] and the seminal books of Simon, [19] and [20], bring us closer to a better understanding of the properties of the zeros of OPUCs. However, there are several open questions about the zeros of OPUCs, see for example pp. 97–98 of [21]. In [11] and [12] the properties of the zeros are studied in terms of analytic properties of the orthogonality measure, while in [21], [22] and [23] the information about the zeros is given in terms of Verblunsky coefficients. Others interesting problems are the description of properties of the zeros of OPUCs in terms of other parameters which also characterize OPUCs. We will deal with some of these questions in this paper.

We need to introduce some notations to state our results. Let  $\mu$  be a non-trivial probability measure on  $[0, 2\pi)$  and let  $\varphi_n(z) = \varphi_n(z, \mu) = \kappa_n z^n + \dots, n = 0, 1, \dots$  denote their orthonormal polynomials with positive leading coefficients,  $\kappa_n > 0$ ,

$$\langle \varphi_n, \varphi_m \rangle = \frac{1}{2\pi} \int \varphi_n(e^{i\theta}) \overline{\varphi_m(e^{i\theta})} d\mu(\theta) = \begin{cases} 1, & \text{si } n = m, \\ 0, & \text{si } n \neq m. \end{cases}$$

Let  $\Phi_n(z) = \frac{\varphi_n(z)}{\kappa_n}$  be the monic OPUC. Then

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \quad n \neq 0 \quad (1)$$

with  $\Phi_0(z) = 1$ . All zeros of  $\Phi_n$  lie in the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  and, therefore,  $\Phi_{n+1}(0) \in \mathbb{D}$ . Moreover, Verblunsky's Theorem (see Theorem 1.7.11, p. 97, of [19]) states that given a sequence  $\{\alpha_n : n = 1, 2, \dots\}$  in  $\mathbb{D}$  there exists a unique probability measure  $\mu$  on  $[0, 2\pi)$  such that  $\Phi(0, \mu) = \alpha_n, n = 1, 2, \dots$

If  $\{z_n\}$  is a sequence in  $\mathbb{D}$  such that

$$\Phi_n(z_n) = 0, \quad n = 1, 2, \dots, \quad (2)$$

then (1) yields

$$\Phi_{n+1}(0) = -z_{n+1} \frac{\Phi_n(z_{n+1})}{\Phi_n^*(z_{n+1})} \quad (3)$$

and Verblunsky's Theorem tells us that there exists a unique orthogonality measure. So, the OPUC are uniquely determined by a sequence of their zeros, i.e., by a sequence  $\{z_n\}$  such that (2) holds.

In this paper we obtain properties of OPUC in term of properties of a sequence of their zeros. For example, we prove the following result about OPUC with periodic zeros.

**Theorem 1.** *Suppose that there exists a common zero for  $\Phi_n$  and  $\Phi_{n-3}$  for all  $n$  large enough. Let  $\zeta_j : j = 1, 2, 3$ , be such common zeros, i.e., there exist  $n_0$  such that for all  $n \geq n_0$ ,*

$$\Phi_n(\zeta_j) = 0, \quad n = j \pmod{3}.$$

$$\text{If } r \stackrel{\text{def}}{=} \max\{|\zeta_j| : j = 1, 2, 3\} \leq \frac{-1+\sqrt{5}}{2},$$

$$\lim_{n \rightarrow \infty} \Phi_n(0) = 0.$$

$$\text{If } r < \frac{-1+\sqrt{5}}{2},$$

$$\limsup_{n \rightarrow \infty} |\Phi_n(0)|^{1/n} \leq \frac{r^2}{1-r} < 1.$$

The numerical experiments show that when the three common zeros have magnitude greater than  $\frac{-1+\sqrt{5}}{2}$ , then the zeros are uniformly distributed on three arcs of unit circle like those corresponding to a measure supported on three arcs, see Figures 3 and 4. If the sequence  $\{z_n\}$  is periodic of period two, then the Verblunsky coefficients are an asymptotic periodic sequence, a situation studied in [3] and [22].

A completely different situation to  $\{z_n\}$  period takes place when it is dense in  $\mathbb{D}$  and so also the zeros of the OPUC are dense in  $\mathbb{D}$ . This case was studied by Khrushchev in [10]. He showed that there exist OPUCs with dense zeros in  $\mathbb{D}$  with orthogonality measure in many classes of measures, including Szegő measures, measures with absolutely convergent series of their Verblunsky parameters (see also Example 1.7.18, p. 98, of [19]). We prove that the zeros can not be “approach the unit circle too fast” if and only if the orthogonality measure lies in the Nevai class.

**Theorem 2.** *The following statements are equivalent*

$$\lim_{n \rightarrow \infty} \Phi_n(0) = 0$$

and

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n (1 - |z_{n,j}|) = \infty,$$

where  $\{z_{n,j} : j = 1, \dots, n\}$  are the zeros of  $\Phi_n$ .

The proof of this theorem is included in Section 5. This section also contains a study of the rate with which the zeros of OPUCs of the Chebyshev weight on an arc of the unit circle approach to  $\partial(\mathbb{D})$ . This example joints the cases studied in [12] of measure with a finite number of singularities in the unit circle and analytic measure corroborate Theorem 2. In Section 2 we make some remarks about general properties of OPUCs in terms of the sequence of zeros stated. In Section 3 we study OPUC with periodic zeros of period two, it is  $\{z_n\}$  has period two. The proof of Theorem 1 is contained in Section 4. We include several figures which display zeros of OPUC for different sequences  $\{z_n\}$  stated.

## 2. General properties

Let  $\{z_n : n = 1, 2, \dots\}$  be a sequence in  $\mathbb{D}$  and let  $\{\Phi_n\}$  be the sequence of monic orthogonal polynomials satisfying (2), i.e.,

$$\Phi_{n+1}(0) = -z_{n+1} \frac{\Phi_n(z_{n+1})}{\Phi_n^*(z_{n+1})} = -z_{n+1} \frac{z_{n+1} - z_n}{1 - \bar{z}_n z_{n+1}} \prod_{j: z_{n,j} \neq z_n} \frac{z_{n+1} - z_{n,j}}{1 - \bar{z}_{n,j} z_{n+1}}.$$

As  $\left| \frac{z_{n+1} - z_{n,j}}{1 - \bar{z}_{n,j} z_{n+1}} \right| < 1$ , the following result follows.

**Lemma 1.** *If  $\lim_{n \rightarrow \infty} \frac{z_{n+1} - z_n}{1 - \bar{z}_n z_{n+1}} = 0$ , then  $\lim_n \Phi_n(0) = 0$ . In particular, if  $\lim z_n = z_0$ ,  $|z_0| < 1$ ,  $\lim_n \Phi_n(0) = 0$ . Moreover, if  $w_n = |z_n - z_0|$ ,*

$$\limsup |\Phi_n(0)|^{1/n} \leq \limsup w_n^{1/n}. \quad (4)$$

*If  $\nu = \lim_{n \rightarrow \infty, n \in \Lambda} \nu_{\Phi_n}^2$ ,  $\lim z_n = z_0$ , and  $\nu(\{z_0\}) = 0$ ,*

$$\limsup_{n \rightarrow \infty, n \in \Lambda} |\Phi_{n+1}(0)|^{1/(n+1)} \leq \exp \int \log \left| \frac{z_0 - \zeta}{1 - \bar{\zeta} z_0} \right| d\nu. \quad (5)$$

*Proof.* We only check (5) because of the other statements are trivial. Let  $\delta > 0$  and  $L = \limsup_{n \rightarrow \infty, n \in \Lambda} |\Phi_{n+1}(0)|^{1/(n+1)}$ . Then

$$\Phi_{n+1}(0) = -z_{n+1} \prod_{j: |z_{n,j} - z_0| < \delta} \frac{z_{n+1} - z_{n,j}}{1 - \bar{z}_{n,j} z_{n+1}} \prod_{j: |z_{n,j} - z_0| \geq \delta} \frac{z_{n+1} - z_{n,j}}{1 - \bar{z}_{n,j} z_{n+1}}$$

$$\begin{aligned} \Rightarrow \frac{1}{n+1} \log |\Phi_{n+1}(0)| &\leq \frac{1}{n+1} \sum_{j: |z_{n,j} - z_0| \geq \delta} \log \left| \frac{z_{n+1} - z_{n,j}}{1 - \bar{z}_{n,j} z_{n+1}} \right| \\ &= \frac{n}{n+1} \int_{|\zeta - z_0| \geq \delta} \log \left| \frac{z_{n+1} - \zeta}{1 - \bar{\zeta} z_{n+1}} \right| d\nu_{\Phi_n}(\zeta). \end{aligned}$$

Since

$$\lim_n \log \left| \frac{z_{n+1} - \zeta}{1 - \bar{\zeta} z_{n+1}} \right| = \log \left| \frac{z_0 - \zeta}{1 - \bar{\zeta} z_0} \right|$$

uniformly on  $\text{sop}(\nu) \cap \{\zeta : |\zeta - z_0| \geq \delta\}$  and  $\lim_{n \in \Lambda} \nu_{\Phi_n} = \nu$ , we have

$$\log L \leq \int_{|\zeta - z_0| \geq \delta} \log \left| \frac{z_0 - \zeta}{1 - \bar{\zeta} z_0} \right| d\nu(\zeta),$$

since  $\delta > 0$  is arbitrary and  $\left| \frac{z_0 - \zeta}{1 - \bar{\zeta} z_0} \right| \leq 1$  for all  $\zeta : |\zeta| \leq 1$ , the integral above is non-positive and monotone decreasing as function of  $\delta$ . Thus, the proof finishes using hypothesis  $\nu(\{z_0\}) = 0$ .  $\square$

---

${}^2\nu_{\Phi_n} \stackrel{\text{def}}{=} \frac{1}{n} \sum_{j=1}^n \delta_{\{z_{n,j}\}}$ . The limit of a sequence of measures throughout this paper is taken in the weak-\* topology.

**Remarks 1.** 1. In [13] it proved that if  $\lim_{n \rightarrow \infty} \sum_{j=1}^n \Phi_j(0) = 0$  and  $\Lambda$  is an infinite subset of natural numbers such that

$$\lim_{n \rightarrow \infty, n \in \Lambda} |\Phi_n(0)|^{1/n} = \limsup |\Phi_n(0)|^{1/n} \stackrel{\text{def}}{=} L,$$

then

$$\lim_{n \rightarrow \infty, n \in \Lambda} \nu_{\Phi_n} = m_L,$$

where  $m_L$  is the Lebesgue measure on the circle of radius  $L$  (the Mhaskar-Saff circle). From (4),  $\limsup |w_n|^{1/n}$  is a bound of the radius  $L$  where the zeros of the polynomials of degree  $n \in \Lambda$  are uniformly distributed.

2. If  $\limsup |\Phi_n(0)|^{1/n} < 1$ , in [21], Simon proved that the rate of convergence of the zeros to the Nevai-Totik points is geometric. So this rate is slower than the radius of the Mhaskar-Saff circle.
3. If  $L > 0$  and  $|z_0| \leq L$ , then

$$\int \log \left| \frac{z_0 - \zeta}{1 - \bar{\zeta} z_0} \right| dm_L(\zeta) = \log L. \quad (6)$$

Thus, (5) yields

$$\limsup_{n \rightarrow \infty, n \in \Lambda} |\Phi_{n+1}(0)|^{1/(n+1)}$$

less than or equal to the infimum of  $L$  such that  $m_L$  is weak-\* limit of some convergent subsequence  $\{\nu_{\Phi_n}\}$  and

$$\{z : |z| \leq L\} \cap \left( \bigcup_{k=1}^{\infty} \overline{\bigcap_{n=k}^{\infty} \{z : \Phi_n(z) = 0\}} \right) \neq \emptyset.$$

Since  $\Phi_{n+1}(0) = (-1)^{n+1} \prod_j z_{n+1,j}$ , (3) implies

$$\Phi_{n+1}(0)^n = \prod_j \frac{\Phi_n(z_{j,n+1})}{\Phi_n^*(j, z_{n+1})}$$

$$\Leftrightarrow |\Phi_{n+1}(0)|^{1/(n+1)} = \exp \iint \log \left| \frac{z-w}{1-\bar{w}z} \right| d\nu_{\Phi_{n+1}}(z) d\nu_{\Phi_n}(w)$$

So, in addition to (5) we have the following

**Lemma 2.** If  $\nu_1 = \lim_{n \rightarrow \infty, n \in \Lambda} \nu_{\Phi_n}$  and  $\nu_2 = \lim_{n \rightarrow \infty, n \in \Lambda} \nu_{\Phi_{n+1}}$ , then

$$\limsup_{n \rightarrow \infty, n \in \Lambda} |\Phi_{n+1}(0)|^{1/(n+1)} \leq \iint \log \left| \frac{z-w}{1-\bar{w}z} \right| d\nu_2(z) d\nu_1(w). \quad (7)$$

*Proof.* The function  $f(z, w) = \log \left| \frac{z-w}{1-\bar{w}z} \right|$  is non-positive upper semicontinuous in  $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$ , so there is a monotone decreasing sequence of non-positive continuous function  $\{g_m\}$  such that  $f(z, w) = \lim_m g_m(z, w)$  pointwise in  $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$  (see

Theorem 1.1, p.1, in [16]). Thus,

$$\begin{aligned} |\Phi_{n+1}(0)|^{1/(n+1)} &= \exp \iint \log \left| \frac{z-w}{1-\bar{w}z} \right| d\nu_{\Phi_{n+1}}(z) d\nu_{\Phi_n}(w) \\ &\leq \exp \iint g_m(z, w) d\nu_{\Phi_{n+1}}(z) d\nu_{\Phi_n}(w), \end{aligned}$$

and since  $\lim_{n \rightarrow \infty} (\nu_{\Phi_{n+1}} \times \nu_{\Phi_n}) = \nu_2 \times \nu_1$ , by the monotone convergence theorem the conclusion follows immediately.  $\square$

**Remark 1.** According to (6), if  $\limsup_{n \rightarrow \infty} |\Phi_{n+1}(0)|^{1/(n+1)} = L$  and  $\nu_1 = \nu_2 = m_L$ , then (7) becomes an equality.

### 3. Zeros of period two

**Lemma 3.** *If the sequence  $\{z_n\}$  is periodic of period two, i.e.,*

$$z_n = \begin{cases} \alpha_1, & n \text{ odd} \\ \alpha_2, & n \text{ even} \end{cases} \quad (8)$$

with  $\{\alpha_1, \alpha_2\} \subset \mathbb{D}$ , then

$$\Phi_1(0) = -\alpha_1, \quad \Phi_2(0) = -\alpha_2 C_{\alpha_1, \alpha_2}, \quad (9)$$

and for all  $n \geq 3$ ,

$$\Phi_n(0) = (-1)^{n-1} C_{\alpha_1, \alpha_2} \begin{cases} \alpha_1^{(n-1)/2} \alpha_2^{(n-1)/2}, & n \text{ odd}, \\ \alpha_1^{-1+n/2} \alpha_2^{n/2}, & n \text{ even}, \end{cases} \quad (10)$$

where  $C_{\alpha_1, \alpha_2} = \frac{\alpha_2 - \alpha_1}{1 - \bar{\alpha}_1 \alpha_2}$ .

*Proof.* Iterating (1), we obtain

$$\Phi_{n+1}(z) = z \left( z + \overline{\Phi_n(0)} \Phi_{n+1}(0) \right) \Phi_{n-1}(z) + (\Phi_{n+1}(0) + z \Phi_n(0)) \Phi_{n-1}^*(z), \quad (11)$$

$n \geq 2$ . If  $\Phi_{n+1}$  and  $\Phi_{n-1}$  have a common zero,  $\zeta$ , then setting  $z = \zeta$  we get

$$\Phi_{n+1}(0) = -\zeta \Phi_n(0)$$

which proves the lemma.  $\square$

**Remark 2.** If  $\min\{|\alpha_1|, |\alpha_2|\} = 0$  or  $\alpha_1 = \alpha_2$ ,  $\Phi_n(0) = 0$  for all  $n \geq 3$  and

$$\Phi_n(z) = z^{n-2} \Phi_2(z), \quad \forall n \geq 3.$$

Thus, we will assume throughout this section that

$$\min\{|\alpha_1|, |\alpha_2|\} > 0 \quad \text{and} \quad \alpha_1 \neq \alpha_2.$$

In [3] it is studied OPUCs with Verblunsky coefficients satisfying  $\lim_n \Phi_n(0) = 0$  and there exists a natural number  $k$  such that

$$\lim_{n \rightarrow \infty, n=j \bmod k} \frac{\Phi_{n+1}(0)}{\Phi_n(0)} \text{ exists, } \quad j = 1, 2, \dots, k. \quad (12)$$

From Lemma 3, (12) holds when  $\{z_n\}$  is periodic of period two.

In [22] going on OPUCs satisfying (12). There, it is required that there exists  $\Delta \in (0, 1)$  such that

$$\Phi_n(0) = \sum_{j=1}^l C_j b_j^n + O(\Delta b^n) \quad (13)$$

where  $0 \notin \{C_j\}$ ,  $\{b_j\}$  are distinct and  $|b_j| = |b| < 1$ ,  $j = 1, \dots, l$ . In our case, if the sequence  $\{z_n\}$  is periodic of period two, then (13) holds with

$$C_1 = -\frac{C_{\alpha_1, \alpha_2}}{2}(\alpha_1 + \frac{1}{\sqrt{\alpha_1 \alpha_2}}), \quad C_2 = -\frac{C_{\alpha_1, \alpha_2}}{2}(\alpha_1 - \frac{1}{\sqrt{\alpha_1 \alpha_2}})$$

and

$$b_1 = \sqrt{\alpha_1 \alpha_2}, \quad b_2 = -\sqrt{\alpha_1 \alpha_2}.$$

Therefore, all results proved in [22] also hold for OPUCs with two periodic zeros. For example,

**Corollary 3.** *If  $\{z_n\}$  satisfies (8),*

$$\begin{aligned} \lim_k \frac{\Phi_{2k}(z)}{\alpha_1^k \alpha_2^k C_{\alpha_1, \alpha_2}} &= \frac{D(0)D(z)^{-1}}{(\alpha_1 \alpha_2 - z^2)}(z - \alpha_2), \\ \lim_k \frac{\Phi_{2k+1}(z)}{\alpha_1^k \alpha_2^k C_{\alpha_1, \alpha_2}} &= \frac{\alpha_2 D(0)D(z)^{-1}}{(\alpha_1 \alpha_2 - z^2)}(\alpha_1 - z), \end{aligned}$$

*uniformly on each compact subset of  $\{z : |z| < \sqrt{|\alpha_1 \alpha_2|}\}$ , where*

$$D(z) = D(z, \mu) = \exp \left( \frac{1}{2\pi} \int \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(\mu'(e^{i\theta})) d\theta \right) \quad (14)$$

*is the Szegő function. As*

$$\kappa \stackrel{\text{def}}{=} \lim_n \kappa_n = D(0)^{-1} = \prod_{j=1}^{\infty} (1 - |\Phi_j(0)|^2)^{-1/2} < \infty,$$

*the analogous results also hold for orthonormal polynomials.*

In particular, if (8) is satisfied

$$\lim_{n \rightarrow \infty} \frac{\Phi_{n+2}(z)}{\Phi_n(z)} = \alpha_1 \alpha_2$$

uniformly on compact subset of  $\{z : |z| < \sqrt{|\alpha_1\alpha_2|}\} \setminus \{\alpha_1, \alpha_2\}$ . Actually, we have only to delete from the disk  $\{z : |z| < \sqrt{|\alpha_1\alpha_2|}\}$  that value  $\alpha_1$  or  $\alpha_2$  of lower magnitude. This result is proved in [3] under the more general condition (12).

It is worthwhile asymptotic behavior in an annulus about the critical circle  $\{z : |z| = \sqrt{|\alpha_1\alpha_2|}\}$ . It requires a parameter  $\Delta_1$  associated to a fine look at of the Verblunsky coefficients. Doing again the calculations in [22], we obtain  $\Delta_1 = \sqrt{|\alpha_1\alpha_2|}$  and the following result.

**Theorem 4.** *If (8) holds,  $D^{-1}(z)$ ,  $|z| < 1$ , admits a meromorphic extension,  $D_{int}^{-1}$ , to  $\{z : |z| < \frac{1}{\sqrt{|\alpha_1\alpha_2|}}\}$  with exactly two poles at  $\pm \frac{1}{\sqrt{\alpha_1\alpha_2}}$  which is analytic in  $\{z : |z| < \frac{1}{\sqrt{|\alpha_1\alpha_2|}}\}$ . Moreover,*

$$\lim_n \Phi_n^*(z) = D(0)D_{int}^{-1}(z)$$

uniformly on compact sets of  $\{z : |z| < \frac{1}{\sqrt{|\alpha_1\alpha_2|}}\}$ . Hence, in  $\{z : |z| > \sqrt{|\alpha_1\alpha_2|}\}$ ,

$$\lim_n \frac{\Phi_n(z)}{z^n} = D(0)\overline{D_{int}(1/\bar{z})}^{-1}.$$

Moreover, let

$$R_n(z) = \Phi_{n+1}(0) (\Phi_n^*(z) - D(0)D^{-1}(z)),$$

and

$$s_n(z) = \sum_{j=0}^{\infty} z^{-j-1} R_{n+j}(z).$$

Then for all  $\epsilon > 0$ ,

$$\max_{|z| \leq 1} |R_n(z)| \leq C((|\alpha_1\alpha_2| + \epsilon)^n),$$

the sums defining each  $s_n$  converge in

$$\mathbb{A} = \{z : |\alpha_1\alpha_2| < |z| < 1\},$$

in this set it defines an analytic function and if  $\epsilon > 0$  is sufficiently small,

$$|s_n(z)| \leq C \frac{(|\alpha_1\alpha_2| + \epsilon)^n}{|z| - |\alpha_1\alpha_2| - \epsilon}.$$

For  $z \in \mathbb{A}$ , we have

$$\Phi_{2k}(z) = s_{2k}(z) + \frac{C_{\alpha_1, \alpha_2} D(0) D(z)^{-1} \alpha_1^k \alpha_2^k}{(\alpha_1 \alpha_2 - z^2)} (z - \alpha_1) + z^{2k} D(0) \overline{D_{int}(1/\bar{z})}^{-1},$$

$$\Phi_{2k+1}(z) = s_{2k+1}(z) + \frac{\beta C_{\alpha_1, \alpha_2} D(0) D(z)^{-1} \alpha_1^k \alpha_2^k}{(\alpha_1 \alpha_2 - z^2)} (\alpha_1 - z) + z^{2k+1} D(0) \overline{D_{int}(1/\bar{z})}^{-1}.$$



**Remark 3.** Using the above result, in [22], Simon also proved that the zeros of OPUCs satisfied what he called “clock behavior” in the Mhaskar-Saff circle  $\{z : |z| = \sqrt{|\alpha_1 \alpha_2|}\}$ : the zeros approach to this circle with rate  $O(\frac{\log n}{n})$ , the rate of magnitudes of consecutive zeros is  $1 + O(\frac{1}{n \log n})$  and their are equally spaced with only a larger gap around  $\pm \sqrt{\alpha_1 \alpha_2}$ . See Figures 1 and 2. These properties of the zeros let us to speculate looking for a justification of what it is seen in the drawings when  $|\alpha_1| \neq |\alpha_2|$ , see Figure 2. For example, if  $|\alpha_2| > |\alpha_1|$ , then  $\alpha_2$  is a Nevai-Totik point, there exists a zero near this point and another one, “*which is not in the gaps*”, accumulates in other point in  $\{z : \sqrt{|\alpha_1 \alpha_2|} < |z| < 1\}$ . Therefore, the Szegő function has two zeros outside this disk which give some equilibrium with its singularities at  $\pm \sqrt{\alpha_1 \alpha_2}$  (remember  $D(\infty) = D(0)^{-1} \neq 0$ ).

### 3.1. On the meromorphic extension of the Szegő function

In Theorem 4 we have required the meromorphic extension of the interior Szegő function to  $\mathbb{A}$ . This function has two poles at  $\pm \sqrt{\alpha\beta}$ . Using technique of Fourier-Padé approximants we prove that an extension with exactly two poles only can be do it to  $\mathbb{A}$ . To obtain that result we use a Lemma stated in [3].

Let  $f \in L^1(\mu)$ . Its Fourier expansion with respect to the orthonormal system  $\{\varphi_n\}$  is given by

$$f(z) \sim \sum_{j=0}^{\infty} A_j \varphi_j(z),$$

where  $A_j$  denotes the Fourier coefficient

$$A_j = \langle f, \varphi_j \rangle.$$

The Fourier-Padé approximant of type  $(n, m)$ ,  $n, m \in \{0, 1, \dots\}$ , of  $f$  is the ratio  $\pi_{n,m}(f) = p_{n,m}/q_{n,m}$  of any two polynomials  $p_{n,m}$  and  $q_{n,m}$  such that

- (i)  $\deg(p_{n,m}) \leq n$ ;  $\deg(q_{n,m}) \leq m$ ,  $q_{n,m} \neq 0$ .
- (ii)  $q_{n,m}(z)f(z) - p_{n,m}(z) \sim A_{n,1}\varphi_{n+m+1}(z) + A_{n,2}\varphi_{n+m+2}(z) + \dots$

Condition (ii) above means that

$$\langle q_{n,m}f - p_{n,m}, \varphi_j \rangle = 0$$

for  $j = 0, \dots, n + m$ . In the sequel, we take  $q_{n,m}$  with leading coefficient equal to 1.

The existence of such polynomials reduces to solving a homogeneous linear system of  $m$  equations on the  $m + 1$  coefficients of  $q_{n,m}$ . Thus a nontrivial solution is guaranteed. In general, the rational function  $\pi_{n,m}$  is not uniquely determined, but if for every solution of (i), (ii), the polynomial  $q_{n,m}$  is of degree  $m$ , then  $\pi_{n,m}$  is unique.

For  $m$  fixed, a sequence of type  $\{\pi_{n,m}, n \in \mathbb{N}\}$ , is called an  $m$ th row of the Fourier-Padé approximants relative to  $f$ . If  $f$  is such that  $R_0(f) > 1$  and has in  $\Delta_m(f)$  exactly  $m$  poles then for all sufficiently large  $n \geq n_0$ ,  $\pi_{n,m}$  is uniquely determined and so is the sequence  $\{\pi_{n,m}, n \geq n_0\}$ . Here  $\Delta_m(f) = \{z : |z| <$

$R_m(f)$  is the largest disk centered at  $z = 0$  in which  $f$  can be extended to a meromorphic function with at most  $m$  poles. This and other results for row sequences of Fourier-Padé approximants may be found in [17], [18] for Fourier expansion with respect to measures supported on an interval of the real line whose absolutely continuous part with respect to Lebesgue's measure is positive almost everywhere.

The following result is stated in [3].

**Lemma 4.** *Let  $\mu$  be such that  $R_0 = R_0(D^{-1}) > 1$ . The following assertions are equivalent:*

- (a)  $D^{-1}$  has exactly  $m$  poles in  $\Delta_m = \Delta_m(D^{-1})$ .
- (b) The sequence  $\{\pi_{n,m}(D^{-1}); n = 0, 1, \dots\}$  for all sufficiently large  $n$  has exactly  $m$  finite poles and there exists a polynomial  $w_m(z) = z^m + \dots$  such that

$$\limsup_n \|q_{n,m} - w_m\|^{1/n} = \delta < 1,$$

where  $\|\cdot\|$  denotes any norm in the space of polynomials of degree at most  $m$ .

The poles of  $D^{-1}$  coincide with the zeros  $z_1, \dots, z_m$  of  $w_m$ , and

$$R_m = \frac{1}{\delta} \max_{1 \leq j \leq m} |z_j|. \quad (15)$$

The lemma above lets us to prove the following result.

**Theorem 5.** *If the sequence of zeros  $\{z_n\}$  satisfies (8), then*

$$R_2(D^{-1}) = \frac{1}{|\alpha\beta|}.$$

To prove the theorem above we need some calculations. Using recurrence relation (1) it is easy to prove the following lemma.

**Lemma 5.**

$$\langle z\varphi_j, 1 \rangle = -\frac{\varphi_{j+1}(0)}{\kappa_j \kappa_{j+1}}.$$

$$\langle z\varphi_j, \varphi_m \rangle = \begin{cases} 0, & \text{si } j < m-1, \\ \frac{\kappa_j}{\kappa_{j+1}}, & \text{si } j = m-1, \\ -\frac{\varphi_{j+1}(0)\overline{\varphi_m(0)}}{\kappa_j \kappa_{j+1}}, & \text{si } j > m-1. \end{cases}$$

For  $j < m-2$ ,

$$\langle z^2\varphi_j, \varphi_m \rangle = 0,$$

and

$$\langle z^2\varphi_{m-2}, \varphi_m \rangle = \frac{\kappa_{m-2}}{\kappa_m}.$$

Moreover,

$$\langle z^2 \varphi_{m-1}, \varphi_m \rangle = -\frac{\overline{\varphi_m(0)}}{\kappa_m^2} \left( \frac{\kappa_{m-1}}{\kappa_{m+1}} \varphi_{m+1}(0) + \frac{\varphi_m(0) \overline{\varphi_{m-1}(0)}}{\varphi_m(0)} \right).$$

If  $j \geq m$ ,

$$\begin{aligned} \langle z^2 \varphi_j, \varphi_m \rangle &= -\frac{\Phi_{j+1}(0) \overline{\varphi_m(0)}}{\kappa_j} \times \\ &\quad \times \left( \frac{\overline{\Phi_{m-1}(0)}}{\overline{\Phi_m(0)}} + \frac{\Phi_{j+2}(0)}{\Phi_{j+1}(0)} - \sum_{l=m-1}^{j+1} \frac{\overline{\Phi_l(0)} \Phi_{l+1}(0)}{\overline{\Phi_m(0)}} \right). \end{aligned}$$

It is known for measures on the Szegő class ( $\log \mu' \in L^1$ ) we have

$$D^{-1}(z) = \frac{1}{\kappa} \sum_{j=0}^{\infty} \overline{\varphi_j(0)} \varphi_j(z), \quad z \in \mathbb{D},$$

see [8], p. 19; [14] Theorem 1; [13] Theorem 2.2; [3], p. 174. If (8) holds, the expansion above converges uniformly on compact subsets of  $\{z : |z| < \frac{1}{\sqrt{|\alpha_1 \alpha_2|}}\}$  and

$$D_{int}^{-1}(z) = \frac{1}{\kappa} \sum_{j=0}^{\infty} \overline{\varphi_j(0)} \varphi_j(z), \quad z \in \{z : |z| < \frac{1}{\sqrt{|\alpha_1 \alpha_2|}}\}. \quad (16)$$

Then, using the Lemma above and (16), we obtain:

**Lemma 6.**

$$\begin{aligned} \langle D_{int}^{-1}, \varphi_m \rangle &= \frac{\overline{\varphi_m(0)}}{\kappa}, \\ \langle z D_{int}^{-1}, \varphi_m \rangle &= \frac{\overline{\varphi_m(0)}}{\kappa} \left( \frac{\overline{\Phi_{m-1}(0)}}{\overline{\Phi_m(0)}} - \sum_{j=m-1}^{\infty} \frac{\overline{\Phi_j(0)} \Phi_{j+1}(0)}{\overline{\Phi_m(0)}} \right), \\ \langle z^2 D_{int}^{-1}, \varphi_m \rangle &= \frac{\overline{\varphi_m(0)}}{\kappa} \left( \frac{\overline{\Phi_{m-2}(0)}}{\overline{\Phi_m(0)}} + O(\varphi_{m-1}(0)) \right). \end{aligned}$$

*Proof.* of Theorem 5. Let us check first that there is not a monic polynomials of degree 1 such that  $q_{n,2}(z) = z - \tau_n$ ; i.e.,

$$\langle q_{n,2} D_{int}^{-1}, \varphi_{n+1} \rangle = \langle q_{n,2} D^{-1}, \varphi_{n+2} \rangle = 0.$$

As  $\langle q_{n,2} D_{int}^{-1}, \varphi_{n+1} \rangle = 0$ , we have

$$\tau_n = \frac{\overline{\Phi_n(0)}}{\overline{\Phi_{n+1}(0)}} - \sum_{j=n}^{\infty} \overline{\Phi_j(0)} \Phi_{j+1}(0) \quad (17)$$

Since also  $\langle q_{n,2}D_{int}^{-1}, \varphi_{n+2} \rangle = 0$ , we obtain

$$\langle zD_{int}^{-1}, \varphi_{n+2} \rangle = \tau_n \langle D_{int}^{-1}, \varphi_{n+2} \rangle \quad (18)$$

From Lemma 6, (17) and (18), we get

$$\frac{\overline{\Phi_{n+1}(0)}}{\overline{\Phi_{n+2}(0)}} = \frac{\overline{\Phi_n(0)}}{\overline{\Phi_{n+1}(0)}} - \overline{\Phi_n(0)}\Phi_{n+1}(0)$$

Since  $\{\frac{\overline{\Phi_{n+1}(0)}}{\overline{\Phi_{n+2}(0)}}, \frac{\overline{\Phi_n(0)}}{\overline{\Phi_{n+1}(0)}}\} = \{-1/\overline{\alpha_1}, -1/\overline{\alpha_2}\}$ ,  $\alpha_1 \neq \alpha_2$ , and  $\lim_n \overline{\Phi_n(0)}\Phi_{n+1}(0) = 0$ , the above relation is imposible for  $n$  large enough.

Thus, the denominators,  $q_{n,2}$ , of Fourier-Padé approximants of order  $(n, 2)$  are exactly of degree 2 for  $n$  large enough.

Let  $q_{n,2}(z) = (z - \beta_n)(z - \tau_n) = z^2 - (\beta_n + \tau_n)z + \beta_n\tau_n$ . It satisfies

$$\langle q_{n,2}D_{int}^{-1}, \varphi_{n+1} \rangle = \langle q_{n,2}D_{int}^{-1}, \varphi_{n+2} \rangle = 0.$$

Thus,

$$\begin{aligned} (\beta_n + \tau_n)\langle zD_{int}^{-1}, \varphi_{n+1} \rangle - \beta_n\tau_n\langle D_{int}^{-1}, \varphi_{n+1} \rangle &= \langle z^2D_{int}^{-1}, \varphi_{n+1} \rangle, \\ (\beta_n + \tau_n)\langle zD_{int}^{-1}, \varphi_{n+2} \rangle - \beta_n\tau_n\langle D_{int}^{-1}, \varphi_{n+2} \rangle &= \langle z^2D_{int}^{-1}, \varphi_{n+2} \rangle. \end{aligned}$$

Hence,

$$\begin{aligned} \beta_n + \tau_n &= \frac{\begin{vmatrix} \langle z^2D_{int}^{-1}, \varphi_{n+1} \rangle & -\langle D_{int}^{-1}, \varphi_{n+1} \rangle \\ \langle z^2D_{int}^{-1}, \varphi_{n+2} \rangle & -\langle D_{int}^{-1}, \varphi_{n+2} \rangle \end{vmatrix}}{\begin{vmatrix} \langle zD_{int}^{-1}, \varphi_{n+1} \rangle & -\langle D_{int}^{-1}, \varphi_{n+1} \rangle \\ \langle zD_{int}^{-1}, \varphi_{n+2} \rangle & -\langle D_{int}^{-1}, \varphi_{n+2} \rangle \end{vmatrix}}, \\ \beta_n\tau_n &= \frac{\begin{vmatrix} \langle zD_{int}^{-1}, \varphi_{n+1} \rangle & \langle z^2D_{int}^{-1}, \varphi_{n+1} \rangle \\ \langle zD_{int}^{-1}, \varphi_{n+2} \rangle & \langle z^2D_{int}^{-1}, \varphi_{n+2} \rangle \end{vmatrix}}{\begin{vmatrix} \langle zD_{int}^{-1}, \varphi_{n+1} \rangle & -\langle D_{int}^{-1}, \varphi_{n+1} \rangle \\ \langle zD_{int}^{-1}, \varphi_{n+2} \rangle & -\langle D_{int}^{-1}, \varphi_{n+2} \rangle \end{vmatrix}}. \end{aligned}$$

We have

$$\begin{aligned} &\begin{vmatrix} \langle zD_{int}^{-1}, \varphi_{n+1} \rangle & -\langle D_{int}^{-1}, \varphi_{n+1} \rangle \\ \langle zD_{int}^{-1}, \varphi_{n+2} \rangle & -\langle D_{int}^{-1}, \varphi_{n+2} \rangle \end{vmatrix} \\ &= \frac{\varphi_{n+1}(0)\varphi_{n+2}(0)}{\kappa^2} \left( \frac{\overline{\Phi_{n+1}(0)}}{\overline{\Phi_{n+2}(0)}} - \frac{\overline{\Phi_n(0)}}{\overline{\Phi_{n+1}(0)}} + \overline{\Phi_n(0)}\Phi_{n+1}(0) \right). \quad (19) \end{aligned}$$

Using, lemma above, we obtain there exists  $C \neq 0$  such that

$$\begin{vmatrix} \langle z^2D_{int}^{-1}, \varphi_{n+1} \rangle & -\langle D_{int}^{-1}, \varphi_{n+1} \rangle \\ \langle z^2D_{int}^{-1}, \varphi_{n+2} \rangle & -\langle D_{int}^{-1}, \varphi_{n+2} \rangle \end{vmatrix} = C \frac{\varphi_{n+1}(0)\varphi_{n+2}(0)}{\kappa^2} \varphi_n(0). \quad (20)$$

Combining (19) and (20), we have there is a constant  $C' \neq 0$  such that

$$\beta_n + \tau_n = C' \varphi_n(0)$$

Doing the same calculation for  $\beta_n \tau_n$ , we obtain

$$\beta_n \tau_n = \frac{\frac{\varphi_{n+1}(0)\varphi_{n+2}(0)}{\kappa^2} \left( \frac{\overline{\Phi_n(0)}}{\overline{\Phi_{n+1}(0)}} \frac{\overline{\Phi_n(0)}}{\overline{\Phi_{n+2}(0)}} - \frac{\overline{\Phi_{n-1}(0)}}{\overline{\Phi_n(0)}} \frac{\overline{\Phi_{n-1}(0)}}{\overline{\Phi_{n+1}(0)}} + O(\varphi_n(0)) \right)}{\frac{\varphi_{n+1}(0)\varphi_{n+2}(0)}{\kappa^2} \left( \frac{\overline{\Phi_{n+1}(0)}}{\overline{\Phi_{n+2}(0)}} - \frac{\overline{\Phi_n(0)}}{\overline{\Phi_{n+1}(0)}} + \overline{\Phi_n(0)}\overline{\Phi_{n+1}(0)} \right)}$$

$$\lim_n \beta_n \tau_n = -\frac{1}{\alpha_1 \alpha_2}$$

with geometric convergence with rate  $\sqrt{|\alpha_1 \alpha_2|}$ .

Therefore,

$$\|q_{n,2}(z) - (z^2 - \frac{1}{\alpha_1 \alpha_2})\|^{1/n} = |\alpha_1 \alpha_2|^{1/2},$$

where the norm is anything in the space of polynomials of degree 2. From Lemma 4,

$$R_2(D_{int}^{-1}) = \frac{|z_j|}{|\alpha_1 \alpha_2|^{1/2}} = \frac{1}{|\alpha_1 \alpha_2|}$$

where  $z_j$  are the roots of  $z^2 - \frac{1}{\alpha_1 \alpha_2}$  (both have magnitude  $\frac{1}{|\alpha_1 \alpha_2|^{1/2}}$ ). It is  $\{z : |z| < \frac{1}{|\alpha_1 \alpha_2|}\}$  is the largest disk centered at  $z = 0$  in which  $D_{int}^{-1}(z)$  can be extended to a meromorphic function with at most two poles.  $\square$

**Remark 4.** An alternative proof of Theorem 5 is a Hadamard formula for  $R_m(D^{-1})$  given in [5]. One of such formula is written in term of  $\varphi_{n+k}^{(j)}(0)$ . These can be obtained using Corollary 3, then the unknown expressions

$$c_m = \int e^{-im\theta} \log w(e^{i\theta}) d\theta,$$

appear. The values  $R_m(D^{-1})$ ,  $j = 0, 1, 2$  founded

$$R_0(D^{-1}) = R_1(D^{-1}) = \frac{1}{\sqrt{|\alpha_1 \alpha_2|}}, \quad R_2(D^{-1}) = \frac{1}{|\alpha_1 \alpha_2|},$$

let us to obtain  $c_0$  and  $c_1$ .

**Remark 5.** From the proof we obtain also that the Fourier-Padé approximants of type  $(n, 1)$  of  $D^{-1}$  has exactly a pole at  $\overline{\alpha_1}^{-1}$  or  $\overline{\alpha_2}^{-1}$  according  $n$  is even or odd. Thus, they converge to  $D_{int}^{-1}$  in  $\{z : |z| < \frac{1}{\sqrt{|\alpha_1 \alpha_2|}}\}$ .

#### 4. Zeros of period three

If the sequence of zeros,  $\{z_n\}$ , is periodic of period three, the Verblunsky coefficients are not a geometric progression as one might naively expect. This case is more complex: if the three periodic zeros have magnitudes at most  $\frac{-1+\sqrt{5}}{2}$ ,

then the measure is in the Nevai class, i.e.,  $\lim_{n \rightarrow \infty} \Phi_n(0) = 0$ , while if the periodic zeros have magnitudes greater than  $\frac{-1+\sqrt{5}}{2}$ , some numerical experiments show that, for large degree, the zeros of OPUCs are close to three arcs of the unit circle, so the orthogonality measure should be supported on these arcs. See Figures 3 and 4.

To prove Theorem 1, we need the following lemma whose proof is easy, so we omit it.

**Lemma 7.** *Let  $\{a_k : k \geq 2\}$  be the sequence*

$$a_2 = a_3 = \frac{2r^2}{1+r^2}, \quad a_{n+1} = r \frac{ra_{n-1} + a_n}{1 + ra_{n-1}a_n}, \quad n \geq 3.$$

*The following statements hold:*

- (i)  $a_n \in (0, r)$  for all  $n \geq 2$ .
- (ii) The sequence  $\{a_n\}$  is monotone decreasing.
- (iii) If  $r \in (0, \frac{-1+\sqrt{5}}{2}]$ ,  $\lim a_n = 0$ , while if  $r \in (\frac{-1+\sqrt{5}}{2}, 1]$ ,

$$\lim a_n = \sqrt{r + 1 - \frac{1}{r}}.$$

- (iv) When  $r < \frac{-1+\sqrt{5}}{2}$ ,

$$\frac{a_n}{a_{n-1}} < \frac{a_{n+1}}{a_{n-1}} = r \frac{r + \frac{a_n}{a_{n-1}}}{1 + ra_{n-1}a_n} < r(r+1) < 1.$$

- (v) Let  $H = \limsup \frac{a_n}{a_{n-1}}$ . We have

$$H \leq r(r+H) \Leftrightarrow H \leq \frac{r^2}{1-r} < r(r+1) < 1.$$

*Proof. of Theorem 1.* From (1) and (11), we have

$$\begin{aligned} \Phi_{n+1}(z) &= \\ &= z \left( z \left( z + \overline{\Phi_n(0)} \Phi_{n+1}(0) \right) + (\Phi_{n+1}(0) + z\Phi_n(0)) \overline{\Phi_{n-1}(0)} \right) \Phi_{n-2}(z) + \\ &\quad + \left( z \left( z + \overline{\Phi_n(0)} \Phi_{n+1}(0) \right) \Phi_{n-1}(0) + \Phi_{n+1}(0) + z\Phi_n(0) \right) \Phi_{n-2}^*(z). \end{aligned}$$

If  $\Phi_{n+1}$  and  $\Phi_{n-2}$  have a common zero  $\zeta$ ,

$$\Phi_{n+1}(0) = -\zeta \frac{\zeta \Phi_{n-1}(0) + \Phi_n(0)}{1 + \zeta \Phi_{n-1}(0) \overline{\Phi_n(0)}}.$$

Hence,

$$|\Phi_1(0)| = |\alpha|, \quad |\Phi_2(0)| \leq |\beta| \frac{|\beta| + |\alpha|}{1 + |\alpha||\beta|} \leq \frac{2r^2}{1+r^2}, \quad |\Phi_3(0)| \leq \frac{2r^2}{1+r^2},$$

$$|\Phi_{n+1}(0)| = |\zeta| \left| \frac{\zeta\Phi_{n-1}(0) + \Phi_n(0)}{1 + \zeta\Phi_{n-1}(0)\overline{\Phi_n(0)}} \right| \leq r \frac{r|\Phi_{n-1}(0)| + |\Phi_n(0)|}{1 + r|\Phi_{n-1}(0)||\Phi_n(0)|}.$$

Let  $\{a_n\}$  be as in the lemma above. Thus,

$$|\Phi_{n+1}(0)| \leq |a_n|, \quad n \geq 2.$$

If  $r \in (0, \frac{-1+\sqrt{5}}{2}]$ , according to lemma above,  $\lim a_n = 0$ . Hence,  $\lim \Phi_n(0) = 0$  holds.

If  $r \in (0, \frac{-1+\sqrt{5}}{2})$ ,

$$\limsup |\Phi_n(0)|^{1/n} \leq \limsup |a_n(0)|^{1/n} \leq \limsup \left| \frac{a_n}{a_{n-1}} \right| < \frac{r^2}{1-r} < 1.$$

□

## 5. Zeros' distance from the unit circle

For the proof of Theorem 2, we require some auxiliary results.

**Lemma 8.** *Let  $\Lambda$  denote an infinite subset of the natural numbers. Let*

$$\{V_n(z) = \prod_{j=1}^n (z - v_{n,j}) : n \in \Lambda\}$$

*be a sequence of monic polynomials whose zeros,  $\{v_{n,j}\}$ , all lie in  $\mathbb{D}$  and such that*

$$\lim_{n \rightarrow \infty, n \in \Lambda} \frac{V_n(z)}{V_n^*(z)} = 0,$$

*uniformly on compact subset of  $\mathbb{D}$ . Suppose there are  $z_0 \in \mathbb{D}$  and  $r > 0$  such that  $V_n(z) \neq 0$  for all  $z : |z - z_0| > r$ . Then*

$$\lim_{n \rightarrow \infty, n \in \Lambda} \sum_{j=1}^n (1 - |v_{n,j}|) = \infty.$$

*Proof.* Without loss, we can assume  $z_0 = 0$ . In fact, by the simple change of variables

$$z = \frac{\zeta + z_0}{1 + \overline{z_0}\zeta}$$

we obtain

$$\frac{W_n(\zeta)}{W_n^*(\zeta)} = \frac{V_n(\frac{\zeta+z_0}{1+\overline{z_0}\zeta})}{V_n^*(\frac{\zeta+z_0}{1+\overline{z_0}\zeta})},$$

where  $W_n$  is a monic polynomials whose zeros,  $\zeta_{n,j}$ ,  $j = 1, \dots, n$ , lie in  $\mathbb{D}$  and there exists  $\delta > 0$  such that

$$|\zeta_{n,j}| > \delta \quad j = 1, \dots, n.$$

Moreover, we have

$$\lim_{n \rightarrow \infty, n \in \Lambda} \sum_{j=1}^n (1 - |v_{n,j}|) = \infty \Leftrightarrow \lim_{n \rightarrow \infty, n \in \Lambda} \sum_{j=1}^n (1 - |\zeta_{n,j}|)$$

according to there are  $k_1 = k_1(z_0) > 0$ ,  $k_2 = k_2(z_0) > 0$  such that

$$k_1(1 - |\zeta|) \leq 1 - \left| \frac{\zeta + z_0}{1 + \overline{z_0}\zeta} \right| \leq k_2(1 - |\zeta|), \quad \forall \zeta \in \mathbb{D}.$$

Thus, we assume  $z_0 = 0$ . By hypothesis,

$$\lim_{n \rightarrow \infty, n \in \Lambda} \prod_{j=1}^n v_{n,j} = \lim_{n \rightarrow \infty, n \in \Lambda} \frac{V_n(0)}{V_n^*(0)} = 0 \Leftrightarrow \lim_{n \rightarrow \infty, n \in \Lambda} \sum_{j=1}^n \log |v_{n,j}| = -\infty. \quad (21)$$

Since  $|v_{n,j}| \geq r$  and there is  $\alpha < -1$  such that  $\alpha x < \log(1 - x)$ ,  $\forall x \in (0, 1 - r)$ , we have

$$\alpha(1 - |v_{n,j}|) < \log(1 - (1 - |v_{n,j}|)) = \log |v_{n,j}|$$

and the proof of the lemma follows from (21).  $\square$

**Remark 6.** A sequence of monic polynomials  $\{V_n(z) = \prod(z - v_{n,j}), n = 1, 2, \dots\}$ , which zeros lie in  $\mathbb{D}$  satisfying

$$\lim_{n \rightarrow \infty} \frac{V_n(z)}{V_n^*(z)} = 0 \quad (22)$$

uniformly on compact subset of  $\mathbb{D}$ , plays a key role in rational approximation. In fact, condition (22) is equivalent to the set of rational functions

$$\left\{ \frac{p_n}{V_n^*} : p_n \text{ polynomial of degree } \leq n, n = 1, 2, \dots \right\}$$

is dense in the space of analytic function in  $\overline{\mathbb{D}}$  with the uniform norm. See Corollary 2, p. 246, in [24].

The next result is a generalization of a Walsh's Theorem (see Theorem 9, p. 247, in [24]).

**Lemma 9.** *Let  $\Lambda$  denote an infinite subset of the natural numbers and let*

$$\{V_n(z) = \prod_{j=1}^n (z - z_{n,j}) : n \in \Lambda\}$$

*be a sequence of monic polynomials whose zeros,  $\{v_{n,j}\}$ , lie in  $\mathbb{D}$ . The following statements are equivalent:*

$$\lim_{n \rightarrow \infty, n \in \Lambda} \frac{V_n(z)}{V_n^*(z)} = 0, \quad (23)$$



uniformly on compact subset of  $\mathbb{D}$ .

$$\lim_{n \rightarrow \infty, n \in \Lambda} \sum_{j=1}^n (1 - |v_{n,j}|) = \infty. \quad (24)$$

*Proof.* Assume (24) holds. Let  $T \in (0, 1)$  be fixed. We have

$$\frac{1-T}{T+1}(1 - |v_{n,j}|) \leq \frac{(1-T)(1 - |v_{n,j}|)}{1 + T|v_{n,j}|} \leq \frac{1-T}{T}(1 - |v_{n,j}|).$$

Thus, (24) is equivalent to

$$\lim_{n \rightarrow \infty, n \in \Lambda} \sum_{j=1}^n \frac{(1-T)(1 - |v_{n,j}|)}{1 + T|v_{n,j}|} = \infty$$

for each  $T \in (0, 1)$ . As  $\frac{(1-T)(1 - |v_{n,j}|)}{1 + T|v_{n,j}|} < 1 - T < 1$ , there exists  $\lambda < -1$  such that

$$\begin{aligned} \lambda \left( \frac{(1-T)(1 - |v_{n,j}|)}{1 + T|v_{n,j}|} \right) &\leq \log \left( 1 - \frac{(1-T)(1 - |v_{n,j}|)}{1 + T|v_{n,j}|} \right) \\ &\leq - \left( \frac{(1-T)(1 - |v_{n,j}|)}{1 + T|v_{n,j}|} \right) \\ \Leftrightarrow \lambda \left( \frac{(1-T)(1 - |v_{n,j}|)}{1 + T|v_{n,j}|} \right) &\leq \log \left( \frac{T + |v_{n,j}|}{1 + T|v_{n,j}|} \right) \leq - \left( \frac{(1-T)(1 - |v_{n,j}|)}{1 + T|v_{n,j}|} \right). \end{aligned}$$

Hence, (24) is equivalent to

$$\lim_{n \rightarrow \infty, n \in \Lambda} \sum_{j=1}^n \log \left| \frac{T + |v_{n,j}|}{1 + T|v_{n,j}|} \right| = -\infty.$$

If  $|z| \leq T$ , we have

$$\left| \frac{V_n(z)}{V_n^*(z)} \right| \leq \prod_{j=1}^n \frac{T + |v_{n,j}|}{1 + T|v_{n,j}|}.$$

Therefore, if (24) holds, then we have (23).

According to Lemma above, to prove (23) implies (24), we need only show the following statement: Assume (23) holds and that for all infinite set  $\Lambda_1 \subset \Lambda$ , all  $z_0 \in \mathbb{D}$ , and for any  $\epsilon > 0$  there exists an infinity set  $\Lambda_2 \subset \Lambda_1$  such that for any  $n \in \Lambda_2$  there is  $j \in \{1, \dots, n\}$  such that  $|v_{n,j}| < \epsilon$ , i.e.,  $V_n$  has a zero in  $\{z : |z - z_0| < \epsilon\}$ , then

$$\lim_{n \rightarrow \infty, n \in \Lambda} \sum_{j=1}^n (1 - |v_{n,j}|) = \infty.$$

To get a contradiction, we assume that there exist  $M > 0$  and an infinite set  $\Gamma \subset \Lambda$  such that

$$\sum_{j=1}^n (1 - |v_{n,j}|) \leq M, \quad \forall n \in \Gamma. \quad (25)$$

We can choose  $w_1, \dots, w_k$  in the circle  $\{z : |z| = 1/2\}$  and  $r > 0$  sufficiently small such that the disks  $\{z : |z - w_j| < r\}$ ,  $j = 1, \dots, k$  are disjoint and

$$k(1/2 - r) > M.$$

According to our assumptions, we can choose an infinity set  $\Gamma_1 \subset \Gamma$  such that

$$V_n(z) \text{ has a zero in } \{z : |z - w_1| < r\} \text{ for all } n \in \Gamma_1.$$

Given  $\Gamma_1$ , we can choose  $\Gamma_2 \subset \Gamma_1 \subset \Gamma$  such that

$$V_n(z) \text{ has a zero in } \{z : |z - w_2| < r\} \text{ for all } n \in \Gamma_2,$$

so,  $V_n$ ,  $n \in \Gamma_2$ , has a zero in  $\{z : |z - w_2| < r\}$  and in  $\{z : |z - w_1| < r\}$ . In this way, there exists an infinity set  $\Gamma_k \subset \Gamma$  such that

$$V_n(z) \text{ has a zero in } |z - w_j| < r \text{ for all } n \in \Gamma_k \text{ and } j = 1, \dots, k.$$

But, just as we have been chosen  $w_1, \dots, w_k$  and  $r$ , we get a contradiction. In fact, because  $\Gamma_k \subset \Gamma$  and for  $n \in \Gamma_k$

$$M \geq \sum_{j=1}^n (1 - |v_{n,j}|) \geq \sum_{j: |v_{n,j} - w_l| < r, l=1, \dots, k} (1 - |v_{n,j}|) > k(1/2 - r) > M.$$

□

*Proof. of Theorem 2.* It is very well known that  $\lim \Phi_n(0) = 0$  is equivalent to

$$\lim_{n \rightarrow \infty} \frac{\Phi_n(z)}{\Phi_n^*(z)} = 0,$$

uniformly on compact subset of  $\mathbb{D}$ , see, for example, Theorem 1.7.4, p. 91 in [19]. Therefore, Theorem 2 follows immediately from Lemma 9.

□

### 5.1. Zero's distance on an arc of the unit circle

The paper [12] proved

$$|z_{n,j}| = 1 - \frac{\log n}{n} + O\left(\frac{1}{n}\right).$$

for the zeros of OPUCs whose orthogonality measure is  $d\mu(\theta) = W(e^{i\theta})d\theta$ , with  $W(z) = w(z) \prod_{k=1}^m |z - a_k|^{2\beta_k}$ ,  $|z| = 1$ ,  $|a_k| = 1$ ,  $\beta_k > -1/2$ ,  $k = 1, \dots, m$ ,

where  $w(z) > 0$ ,  $|z| = 1$ , has analytic continuation to an annulus around the unit circle.

It is known that for positive weight almost everywhere on an arc,  $\Delta$ , of the unit circle, the zeros of their orthogonal polynomials approach to the unit circle (see [6]) as the degree of the polynomials increasing. Moreover, see [7], on each neighborhood of each arc,  $\Delta' \subset \Delta$ , there exist  $O(n)$  zeros of  $\Phi_n$  for  $n$  large enough.

Next, we find the rate with which the zeros of OPUCs of the Chebyshev weight on an arc of the unit circle approach to  $\partial(\mathbb{D})$ . Consider the weight

$$w(\theta) = \frac{\sin(\alpha/2)}{2 \sin(\theta/2) \sqrt{\cos^2 \alpha/2 - \cos^2 \theta/2}}, \quad \theta \in [\alpha, 2\pi - \alpha],$$

let  $\{\Phi_n\}$  be the sequence of monic orthogonal polynomial for  $w$  and  $\{z_{n,j} : j = 1, \dots, n\}$  are their zeros.

**Theorem 6.** *If  $\lim_{n \rightarrow \infty} z_{n,j_n} = e^{i\theta_0}$  with  $\theta_0 \in [\alpha, 2\pi - \alpha]$ ,*

$$|z_{n,j_n}| = 1 - f(\theta_0)/n + O(1/n^2),$$

where  $f$  is a positive continuous function in  $[\alpha, 2\pi - \alpha]$  which is nonzero in  $(\alpha, 2\pi - \alpha)$ .

*Proof.* We have

$$\varphi_n(z) = K_n \left\{ \frac{w^n(v)}{1 - \beta v} + \frac{v w^n(1/v)}{v - \beta} \right\}, \quad z = h(v), \quad (26)$$

where  $\beta = i \tan \frac{\pi - \alpha}{4}$ ,

$$w(v) = i \frac{1 - \beta v}{v + \beta}, \quad w(1/v) = i \frac{v - \beta}{1 + \beta v}, \quad z = h(u) = \frac{(v - \beta)(\beta v - 1)}{(v + \beta)(\beta v + 1)}.$$

See [9]. Also, these polynomials were studied by Akhiezer.

The function  $w = w(v)$  is an invertible analytic homeomorphism of  $\mathbb{D}$  to  $\mathbb{D}$  and  $z = h(u) = \frac{(v - \beta)(\beta v - 1)}{(v + \beta)(\beta v + 1)}$  is analytic in  $\mathbb{C} \setminus \{-\beta, -\frac{1}{\beta}\}$  and a homeomorphism of  $\mathbb{D} \setminus \{-\beta\}$  to  $\mathbb{C} \setminus \Delta_\alpha$  where,  $\Delta_\alpha = \{e^{i\theta} : \theta \in [\alpha, 2\pi - \alpha]\}$ .

Each zero  $z_{n,j}$  of (26) corresponds with a unique  $v_{n,j} : h(v_{n,j}) = z_{n,j}$ ,  $|v_{n,j}| < 1$ . Since  $z_{n,j} \rightarrow \Delta_\alpha$ , as  $n \rightarrow \infty$ , we have  $|v_{n,j}| \rightarrow 1$ . Moreover, we know

$$\frac{w^n(v_{n,j})}{1 - \beta v_{n,j}} + \frac{v_{n,j} w^n(1/v_{n,j})}{v_{n,j} - \beta} = 0 \Leftrightarrow \frac{w^n(1/v_{n,j})}{w^n(v_{n,j})} = -\frac{v_{n,j} - \beta}{v_{n,j}(1 - \beta v_{n,j})}. \quad (27)$$

Consider  $e^{i\theta_0} \in \Delta_\alpha$  with  $\theta_0 \in (\alpha, 2\pi - \alpha)$  and  $e^{i\theta_0} = \lim_n z_{n,j}$ , here  $j = j_n$  changes with  $n$ . Actually, we have  $n \in \Lambda$  a sequence of indexes such that  $z_{n,j}, n \in \Lambda$ , has limit  $e^{i\theta_0}$ . Throughout, we consider such indexes. Thus,

$$\lim_n v_{n,j} = e^{i\omega_0}, \quad e^{i\theta_0} = h(e^{i\omega_0}), \quad \omega_0 \in (0, \pi),$$

$$\lim_n \Im(v_{n,j}) = \sin \omega_0,$$

$$\lim_n \frac{1}{|v_{n,j}|^2} \frac{|v_{n,j}|^2 - 2\Re(\overline{v_{n,j}}\beta) + |\beta|^2}{1 - 2\Re(v_{n,j}\beta) + |v_{n,j}\beta|^2} = \frac{1 - 2\tan \eta \sin \omega_0 + |\beta|^2}{1 + 2\tan \eta \sin \omega_0 + |\beta|^2} \in (0, 1).$$

On the other hand,

$$\begin{aligned} \left| \frac{v_{n,j} - \beta}{v_{n,j}(1 - \beta v_{n,j})} \right|^2 &= \frac{1}{|v_{n,j}|^2} \frac{(v_{n,j} - \beta)(\overline{v_{n,j}} - \overline{\beta})}{(1 - \beta v_{n,j})(1 - \overline{\beta} \overline{v_{n,j}})} \\ &= \frac{1}{|v_{n,j}|^2} \frac{|v_{n,j}|^2 - 2\Re(\overline{v_{n,j}}\beta) + |\beta|^2}{1 - 2\Re(v_{n,j}\beta) + |v_{n,j}\beta|^2}, \end{aligned}$$

$$\text{and } \Re(v_{n,j}\beta) = -\Im(v_{n,j}) \tan \eta, \Re(\overline{v_{n,j}}\beta) = \Im(v_{n,j}) \tan \eta.$$

Moreover,

$$|w(1/v_{n,j})|^2 = \left| \frac{1 - \beta v_{n,j}}{v_{n,j} + \beta} \right|^2 = \frac{1 - 2\Re(v_{n,j}\beta) + |v_{n,j}\beta|^2}{|v_{n,j}|^2 + 2\Re(\overline{v_{n,j}}\beta) + |\beta|^2} \rightarrow 1, \quad \text{as } n \rightarrow \infty,$$

$$\begin{aligned} |w(1/v_{n,j})|^2 - 1 &= \frac{(1 - |v_{n,j}|^2)(1 - |\beta|^2)}{1 + 2\Re(\overline{v_{n,j}}\beta) + |\beta|^2} \sim 2(1 - |v_{n,j}|) \frac{(1 - |\beta|^2)}{1 + 2\tan \eta \sin \omega_0 + \tan^2 \eta}. \end{aligned}$$

$$|w(v_{n,j})|^2 = \left| \frac{v_{n,j} - \beta}{1 + \beta v_{n,j}} \right|^2 = \frac{|v_{n,j}|^2 - 2\Re(\overline{v_{n,j}}\beta) + |\beta|^2}{1 + 2\Re(v_{n,j}\beta) + |v_{n,j}\beta|^2} \rightarrow 1, \quad \text{as } n \rightarrow \infty,$$

$$|w(v_{n,j})|^2 - 1 = \frac{(|v_{n,j}|^2 - 1)(1 - |\beta|^2)}{1 + 2\Re(v_{n,j}\beta) + |v_{n,j}\beta|^2} \sim -2(1 - |v_{n,j}|) \frac{(1 - |\beta|^2)}{1 - 2\tan \eta \sin \omega_0 + |\beta|^2}.$$

Hence,

$$\begin{aligned} \left| \frac{w(1/v_{n,j})}{w(v_{n,j})} \right|^2 - 1 &= \frac{|w(1/v_{n,j})|^2 - |w(v_{n,j})|^2}{|w(v_{n,j})|^2} = \frac{-2(1 - |\beta|^2)(1 - |v_{n,j}|)}{|w(v_{n,j})|^2} \\ &\xrightarrow{n \rightarrow \infty} \frac{4\tan \eta \sin \omega_0}{(1 + 2\tan \eta \sin \omega_0 + |\beta|^2)(1 - 2\tan \eta \sin \omega_0 + |\beta|^2)} \\ &= \frac{-8(1 - |\beta|^2)(1 - |v_{n,j}|) \tan \eta \sin \omega_0}{(1 - 2\tan \eta \sin \omega_0 + |\beta|^2)(1 + 2\tan \eta \sin \omega_0 + |\beta|^2)}. \end{aligned}$$

From (27), we obtain

$$\left| \frac{w(1/v_{n,j})}{w(v_{n,j})} \right|^n \xrightarrow{n} \frac{1 - 2\tan \eta \sin \omega_0 + |\beta|^2}{1 + 2\tan \eta \sin \omega_0 + |\beta|^2},$$

so,

$$n \left( \left| \frac{w(1/v_{n,j})}{w(v_{n,j})} \right| - 1 \right) \xrightarrow{n} \log \left( \frac{1 - 2\tan \eta \sin \omega_0 + |\beta|^2}{1 + 2\tan \eta \sin \omega_0 + |\beta|^2} \right),$$

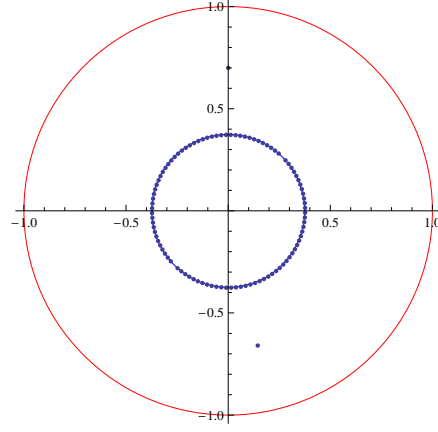


Figure 1: Zeros of  $\Phi_{100}$  for two period zeros: 0.2 and  $0.7i$ .

Thus,

$$\lim_n n(1 - |v_{n,j}|) = f(\omega_0)$$

where

$$f(\omega_0) = \frac{(1 - 2 \tan \eta \sin \omega_0 + |\beta|^2)(1 + 2 \tan \eta \sin \omega_0 + |\beta|^2)}{8(1 - |\beta|^2) \tan \eta \sin \omega_0} \times \log \left( \frac{1 + 2 \tan \eta \sin \omega_0 + |\beta|^2}{1 - 2 \tan \eta \sin \omega_0 + |\beta|^2} \right).$$

Therefore,

$$\begin{aligned} |z_{n,j}| &= |h(v_{n,j})| = |w(v_{n,j})| |w(\frac{1}{v_{n,j}})| \\ &= (1 + |w(v_{n,j})| - 1)(1 + |w(\frac{1}{v_{n,j}})| - 1) \\ &\sim \left( 1 + \frac{2(1 - |v_{n,j}|)(1 - |\beta|^2)}{1 + 2 \tan \eta \sin \omega_0 + |\beta|^2} \right) \left( 1 + \frac{2(|v_{n,j}| - 1)(1 - |\beta|^2)}{1 - 2 \tan \eta \sin \omega_0 + \tan^2 \eta} \right) \\ &\sim 1 - \frac{1}{n} \tilde{f}(\omega_0), \end{aligned}$$

where

$$\tilde{f}(\omega_0) = \log \left( \frac{1 + 2 \tan \eta \sin \omega_0 + \tan^2 \eta}{1 - 2 \tan \eta \sin \omega_0 + \tan^2 \eta} \right).$$

□

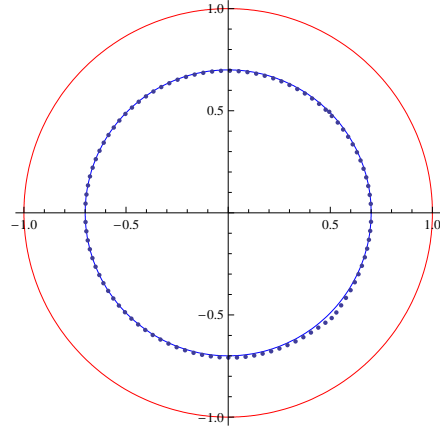


Figure 2: Zeros of  $\Phi_{100}$  for two period zeros:  $0.7e^{-i\frac{\pi}{4}}$  and  $0.7e^{i\frac{\pi}{4}}$ .

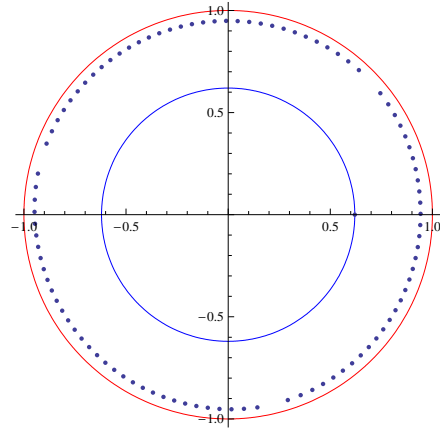


Figure 3: Zeros of  $\Phi_{100}$  for three period zeros:  $0.62$ ,  $0.62e^{i\frac{2\pi}{3}}$  and  $0.62e^{-i\frac{2\pi}{3}}$ . Observe  $\frac{-1+\sqrt{5}}{2} = 0.618034\dots < 0.62$

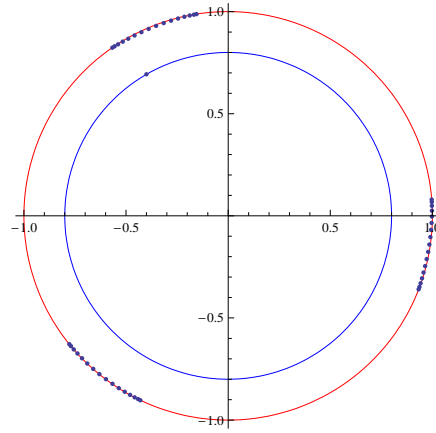


Figure 4: Zeros of  $\Phi_{50}$  for three period zeros:  $0.8$ ,  $0.8e^{i\frac{2\pi}{3}}$  and  $0.8e^{-i\frac{2\pi}{3}}$ . When the degree of the OPUC is larger than 50 calculating the zeros appear numerical instability in Mathematica

**Remark 7.** The Figures 1–4 were generated in Mathematica 6.

## References

- [1] *Alfaro, M. P.; Vigil, L.* Solution of a problem of P. Turán on zeros of orthogonal polynomials on the unit circle. *J. Approx. Theory* 53 (1988), no. 2, 195–197.
- [2] *Alfaro, M. P.; Bello-Hernández, M.; Montaner, J. M.; Varona, J. L.* , Some asymptotic properties for orthogonal polynomials with respect to varying measures. *J. Approx. Theory* 135 (2005), no. 1 , 22–34.
- [3] *Barrios Rolana, D.; Lpez Lagomasino, G.; Saff, E.B.* Asymptotics of orthogonal polynomials inside the unit circle and Szegő-Padé approximants. *J. Comp. Appl. Math.* 133 (2001), no. 1–2, 171–181.
- [4] *Barrios Rolana, D.; Lpez Lagomasino, G.; Saff, E.B.* Determining radii of meromorphy via orthogonal polynomials on the unit circle. *J. Approx. Theory* 124 (2003), no. 2, 263–281
- [5] *Barrios Rolana, D.; Lpez Lagomasino, G.; Saff, E.B.* Determining radii of meromorphy via orthogonal polynomials on the unit circle. *J. Approx. Theory* 124 (2003), no. 2, 263–281.
- [6] *Bello Hernandez, M.; Lpez Lagomasino, G.* Ratio and relative asymptotics of polynomials orthogonal on an arc of the unit circle. *J. Approx. Theory* 92 (1998), no. 2, 216–244.

- [7] *Bello Hernandez, M.; Mia Daz, E.* Strong asymptotic behavior and weak convergence of polynomials orthogonal on an arc of the unit circle. *J. Approx. Theory* 111 (2001), no. 2, 233–255.
- [8] *Geronimus, L. Ya.* Orthogonal Polynomials, Consultants Bureau, New York, 1961.
- [9] *Golinskii, L.* Akhiezer’s orthogonal polynomials and Bernstein-Szego” method for a circular arc. *J. Approx. Theory* 95 (1998), no. 2, 229–263.
- [10] *Khrushchev, S.* Turn measures. *J. Approx. Theory* 122 (2003), no. 1, 112–120.
- [11] *Martnez-Finkelshtein, A.; McLaughlin, K. T.-R.; Saff, E. B.* Szegő orthogonal polynomials with respect to an analytic weight: canonical representation and strong asymptotics. *Constr. Approx.* 24 (2006), no. 3, 319–363.
- [12] *Martnez-Finkelshtein, A.; McLaughlin, K. T.-R.; Saff, E. B.* Asymptotics of orthogonal polynomials with respect to an analytic weight with algebraic singularities on the circle. *Int. Math. Res. Not.* 2006, Art. ID 91426, 43 pp.
- [13] *Mhaskar, H. N.; Saff, E. B.* On the distribution of zeros of polynomials orthogonal on the unit circle. *J. Approx. Theory* 63 (1990), no. 1, 30–38.
- [14] *Nevai, P.; Totik, V.* Orthogonal polynomials and their zeros. *Acta Sci. Math. (Szeged)* 53 (1989), no. 1–2, 99–104.
- [15] *Peherstorfer, F.; Steinbauer, R.* Asymptotic behaviour of orthogonal polynomials on the unit circle with asymptotically periodic reflection coefficients. *J. Approx. Theory* 88 (1997), no. 3, 316–353.
- [16] *Saff, E. B.; Totik, V.*s Logarithmic potentials with external fields. Appendix B by Thomas Bloom. *Grundlehren der Mathematischen Wissenschaften*, 316. Springer-Verlag, Berlin, 1997.
- [17] *Suetin, S.P.* On the convergence of rational approximations to polynomials expansions in domains of meromorphy of a given function, *Mat. Sb.* 105(147) (1978) 413–430 (English transl. in *Math USSR Sb.* 34 (1978) 367–381).
- [18] *Suetin, S.P.* Inverse theorems on generalized Padé approximants, *Mat. Sb.* 109(151) (1979) 629–646 (English transl. in *Math. USSR Sb.* 37(4) (1980) 581–597).
- [19] *Simon, B.* Orthogonal polynomials on the unit circle. Part 1. Classical theory. *AMS Colloquium Publications*, 54, Part 1. AMS, Providence, RI, 2005.
- [20] *Simon, B.* Orthogonal polynomials on the unit circle. Part 2. Classical theory. *AMS Colloquium Publications*, 54, Part 1. AMS, Providence, RI, 2005.



- [21] *Simon, B.* Fine structure of the zeros of orthogonal polynomials. I. A tale of two pictures. *Electron. Trans. Numer. Anal.* 25 (2006), 328–368.
- [22] *Simon, B.* Fine structure of the zeros of orthogonal polynomials. II. OPUC with competing exponential decay. *J. Approx. Theory* 135 (2005), no. 1, 125–139.
- [23] *Simon, B.* Fine structure of the zeros of orthogonal polynomials. III. Periodic recursion coefficients. *Comm. Pure Appl. Math.* 59 (2006), no. 7, 1042–1062.
- [24] *Walsh, J. L.* *Interpolation and Approximation by Rational Functions in the Complex Domain.* AMS Colloquium Publications, XX, Third Ed. 1960.